Plasticity and Deformation Process

Strain calculations in plastic deformation

Solution of Plastic Deformation Problems

We combine the yield criterion, the stress-strain relations, and the material model to solve for the deformations in a plastic deformation process utilizing proportional loading

Recall that the generally applicable yield criterion for plastic materials is von Mises':

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)} = \sigma_y$$

The multiaxial stress state should be translated into the effective stress and compared with the yield strength of the material to find the plastic deformation and the dynamic material properties that are dependent on the plastic strains

$$E_{sec} = \frac{\sigma_{eff}}{\varepsilon_{eff}}$$

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

$$G^p = \frac{E_{sec}}{2(1+\nu)}$$

Procedure to obtain the secant modulus form a best fit curve representing the uniaxial stress-strain curve:
Basically the curve should be considered as a set of points each of which is a stress and corresponding strain.

- Calculate value of σ_{eff} corresponding to the given multiaxial stress state
- Determine E_{sec} analytically by interpolation of the stress values from a table of stress-strain data pairs or from the best fit curve equation.
- Use E_{sec} and the variable ν in the stress-strain relations to calculate the strains for the specified σ_{eff}
- The strains $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ that are given below are the answers we are looking for, not ε_{eff} that can be obtained directly from the curve.

$$\varepsilon_x = \frac{1}{E_{sec}}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

$$\varepsilon_y = \frac{1}{E_{sec}}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\varepsilon_z = \frac{1}{E_{sec}}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G^p}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G^p}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G^p}$$

Material Model Equations

Material model equations enable analytic calculation of the secant modulus using a best fit curve to the uniaxial stress-strain diagram

There are five model equations to represent the stress-strain curves of common strain-hardening materials:

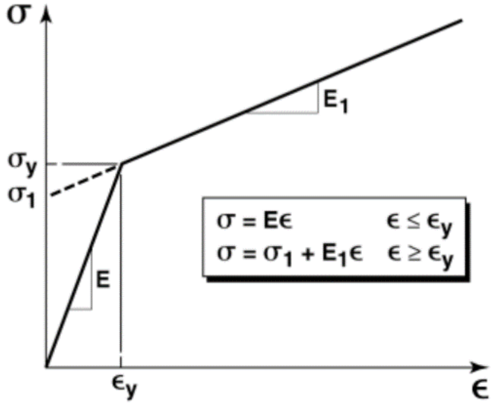
1. Linear strain hardening
2. Power-law
3. Ramberg-Osgood
4. Nadai
5. Nadai-Jones

These models cover more than one class of strain-hardening materials in addition to the non-strain hardening elastic-perfectly plastic and rigid-perfectly plastic curves.

The least number of parameters to describe elastic-perfectly plastic stress-strain behavior is 2: E and σ_y

At least three parameters will be needed to approximate non-linear stress-strain behavior using these models.

1. Linear strain-hardening model is the simplest of all non-linear stress-strain curve models. It consists of two straight lines with different moduli:



$$\sigma = E\varepsilon \qquad \varepsilon \le \varepsilon_y$$
$$\sigma = \sigma_1 + E_1\varepsilon \qquad \varepsilon > \varepsilon_y$$

It becomes the elastic-perfectly plastic model when $E_1 = 0$ and $\sigma_1 = \sigma_y$,
and the linear elastic model when $E_1 = E$ and $\sigma_1 = 0$

Three parameters are necessary to determine the strains as functions of stress: $E \;\; , E_1, \sigma_y$

The elastic and the first moduli are determined graphically from the secant modulus:

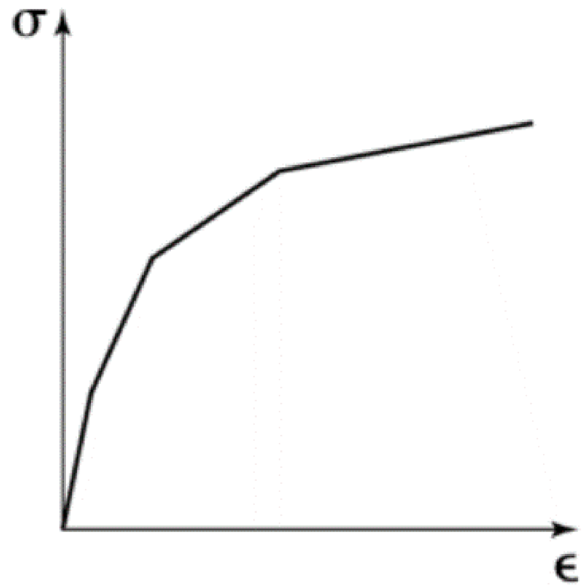
$$E_{sec} = E \qquad \sigma \le \sigma_y$$
$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma \ge \sigma_y$$

The stress at yielding is written by the use of the equation of a straight line y=mx+b:

$$\sigma_y = \sigma_1 + E_1\varepsilon_y$$

The stress-strain behavior of many materials can be represented roughly with this model.

Philip’s model is a mathematical extension of the linear strain-hardening model which consists of multiple straight line segments.



$$E_{sec} = E \qquad \sigma \leq \sigma_y$$

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma \geq \sigma_y$$

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_2 \qquad \sigma \geq \sigma_m$$

$$\ldots$$

The more line segments that exist, the better the measured stress-strain behavior can be modeled.
However the mathematical difficulty increases with the addition of each line segment as more parameters are introduced in the model equation.

2. Power-law model

The general form of power-law stress-strain curve model has the following equation

$$\sigma = A\varepsilon^n$$

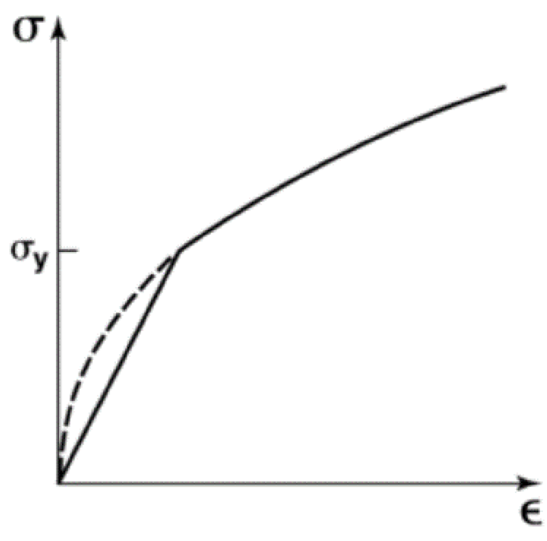
n is the strain hardening coefficient, A is the constant which are adjusted to best fit measured stress-strain data.

The value of n should be in the range 0-1 in order to model concave-downward behavior.

The stress-strain curve has an infinite slop at the origin and the equation is not good for low stress levels. Instead the following form is used to account for elastic deformations:

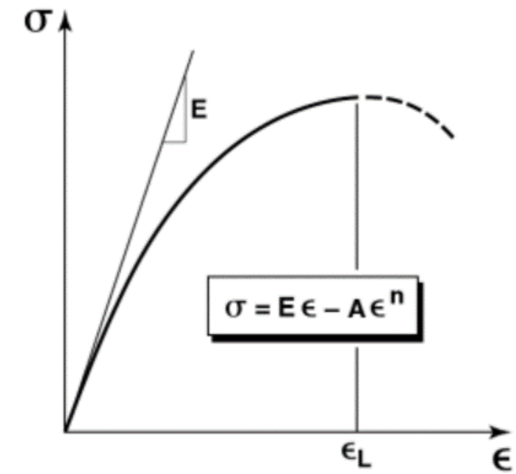
$$\sigma = E\varepsilon \qquad \sigma \le \sigma_y$$

$$\sigma = A\varepsilon^n \qquad \sigma \ge \sigma_y$$



An alternative form is

$$\sigma = E\varepsilon - A\varepsilon^{n}$$



Only three stress-strain curve parameters are needed for this equation: E, A, n

It is valid until the maximum stress-strain point corresponding to ε_L

$$\varepsilon_L = \left(\frac{E}{An}\right)^{\frac{1}{n-1}}$$

Power-law model is used extensively because of its mathematical simplicity, however only certain types of stress-strain behavior can be modeled with it.

3. Ramberg-Osgood Model

The general form of Ramberg-Osgood stress-strain curve equation is

$$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{H}\right)^n$$

The first part in the right hand side is the elastic strain and the second is the plastic strain

K is a constant depending on the plastic modulus H and n

n is the inverse of strain hardening coefficient and it is found from two data points from the non-linear part of the stress-strain diagram:

$$\frac{1}{n} = \frac{\log(\sigma_2/\sigma_1)}{\log(\varepsilon_2/\varepsilon_1)}$$

H, the plastic modulus is obtained again from a stress-strain relationship at the non-linear part:

$$H = \frac{\sigma_1}{{\varepsilon_1}^{1/n}}$$

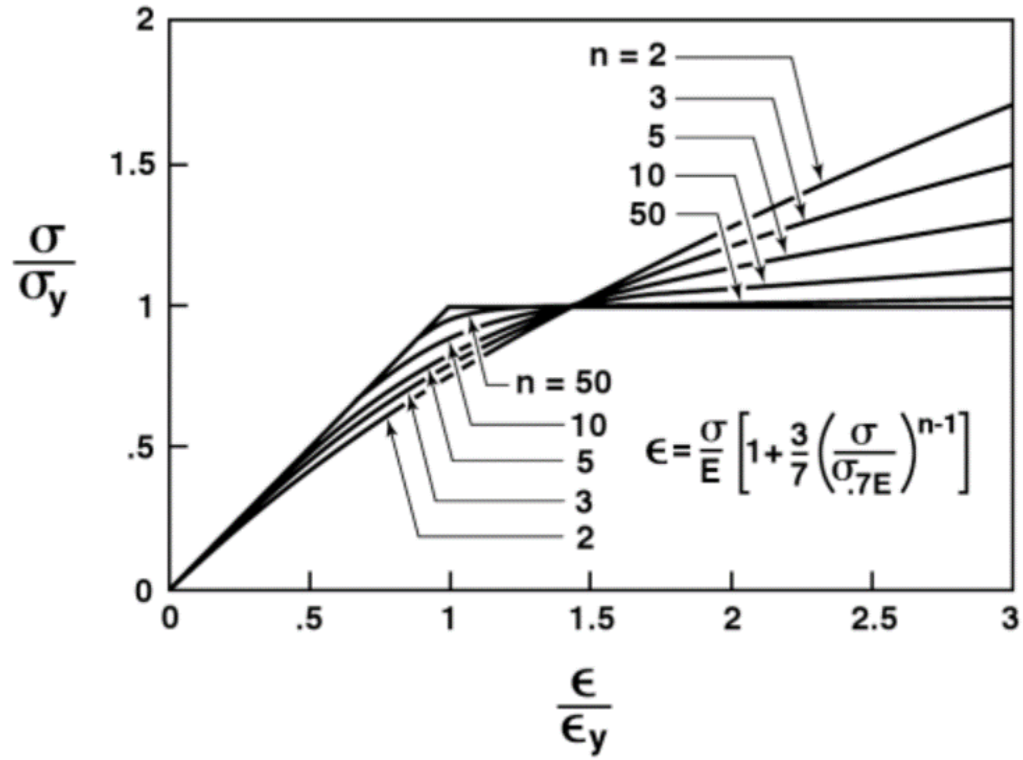
The equation becomes the following form when we take the off-set yield point as our data point

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{\sigma_y}\right)^n$$

n can also be determined by iteration during curve-ftting

$$\varepsilon = \frac{\sigma}{E}\left(1 + 0.002\left(\frac{\sigma}{\sigma_y}\right)^{n-1}\right)$$

Three parameters are needed to determine the Ramberg-Osgood stress-strain curve: E, σ_y and n



The model equation is continuously curved, there is no definitive elastic region followed by a yield stress. It approaches elastic-perfectly plastic behavior as n gets larger

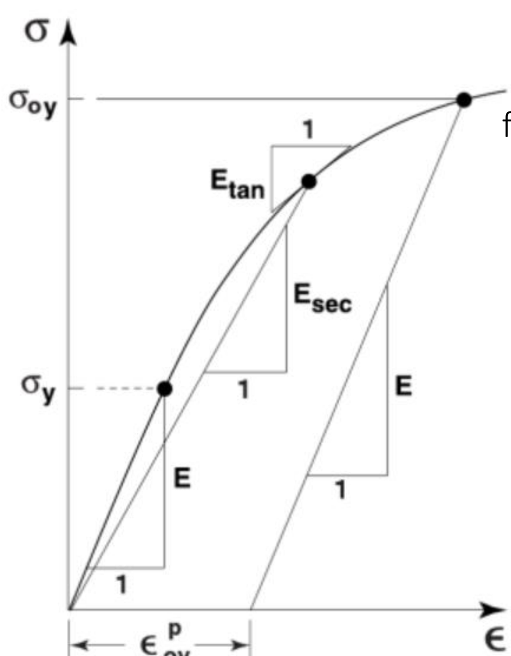
4. Nadai Model

The behavior of elastic-plastic materials like aluminum and its alloys are represented well with a linear elastic region ended by a well defined yield stress and a gradual bending over of the concave downward stress-strain curve.
Nadai model equation represents this behavior:

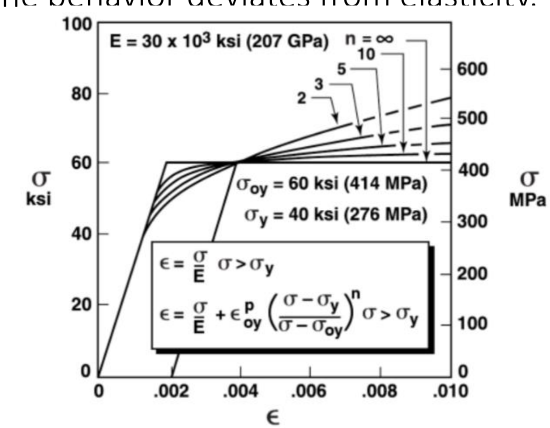
$$\varepsilon = \frac{\sigma}{E} \qquad \sigma \le \sigma_y$$
$$\varepsilon = \frac{\sigma}{E} + K\left(\sigma - \sigma_y\right)^n \qquad \sigma \ge \sigma_y$$

Where K is constant dependent on the fitting parameter n, the off-set yield strain $\varepsilon_{oy} = 0.002$, and the stress at the off-set yield point:

$$K = {\varepsilon_{oy}}^p \left(\sigma - \sigma_{oy}\right)^{-n}$$



The off-set yield strain at 0.002 is determined from the permanent strain for materials like steel and aluminum where the behavior deviates from elasticity.



$$\varepsilon = \frac{\sigma}{E} + {\varepsilon_{oy}}^p \left(\frac{\left(\sigma - \sigma_y\right)}{\left(\sigma - \sigma_{oy}\right)}\right)^n$$

Nadai model needs four parameters above the yield stress: E, σ_y, σ_{oy}, n
The model equation is similar to the Ramberg-Osgood equation

5. Nadai-Jones Model

The concept of Nadai stress-strain curve model is extended to cover plastic materials with two distinctly different regions of nonlinear behavior. Nadai-Jones equation is the same until an upper stress where a second highly nonlinear region is reached:

$$\varepsilon = \frac{\sigma}{E} \qquad \sigma \le \sigma_y$$

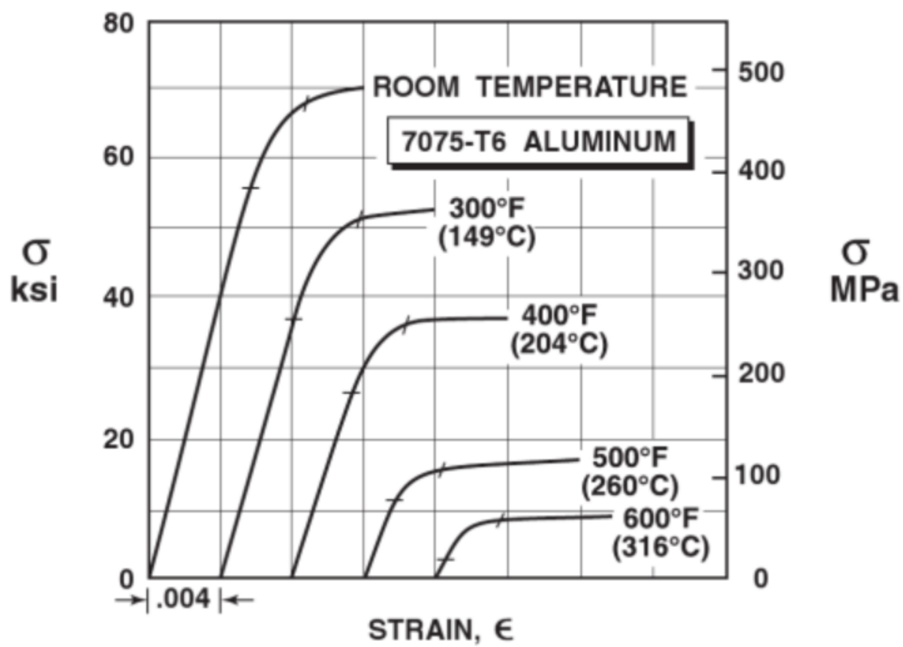
$$\varepsilon = \frac{\sigma}{E} + K\left(\sigma - \sigma_y\right)^n \qquad \sigma_2 \ge \sigma \ge \sigma_y$$

$$\varepsilon = \frac{\sigma}{E} + K\left(\sigma - \sigma_y\right)^n + J(\sigma - \sigma_2)^m \qquad \sigma_3 \ge \sigma \ge \sigma_2$$

Where K and J are constants that depend on upper stresses σ_2 and σ_3 ,the corresponding plastic strains ${\varepsilon_2}^p$ and ${\varepsilon_3}^p$and the curve fitting constants n and m

$$K = {\varepsilon_2}^p\left(\sigma_2 - \sigma_y\right)^{-n}$$
$$J = {\varepsilon_3}^p\left(\sigma_3 - \sigma_y\right)^{-n}$$

Example - Analyze the uniaxial deformation of an aluminum alloy using different models and determine the strains when it is loaded in plane stress condition with normal stresses in the x and y direction of 70 MPa and 40 MPa, and with shearing stress of 30 MPa



Compressive Stress-Strain Curves for 7075-T6 Aluminum as a Function of Temperature *(After Mathauser and Brooks*

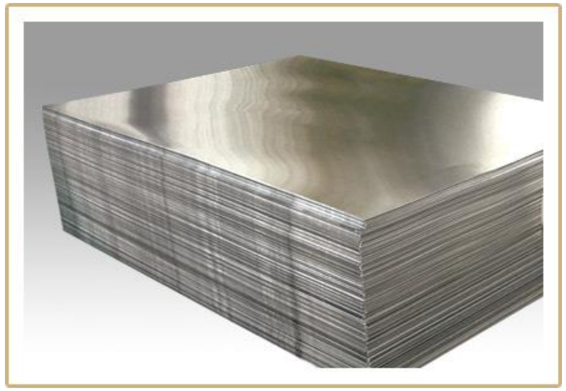
Example – Nickel deforms plastically above a yield strength of 185 MPa. Its E=207 GPa and $\nu = 0.31$. The deformation of the material can be represented by power law model with A= 700, n=0.4.
Obtain the secant modulus using the power law model for an effective stress of 500 MPa.

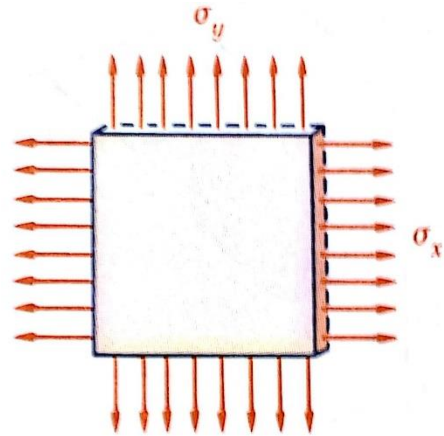
σ=$A\varepsilon^{n}$

E_{sec}=(σ_y+$A\varepsilon^{n}$)/ ε_T

$$\varepsilon_T = \sqrt[n]{\left(\frac{\sigma_{eff} - \sigma_y}{A}\right)} + \varepsilon_y$$

Case Study - Consider an aluminum sheet with sides 1 meter long and that is negligibly thin in the z direction so that plane stress condition is valid





The loads applied to the material are tensile stress in the x direction and a shear stress, tensile stresses in both axes and shear in the plane.

Let's apply the deformation theory and use three material models linear hardening, power law, ramberg-osgood models, to find the strains in each case

Linear hardening equations $\sigma = \sigma_1 + E_1\varepsilon$ $\qquad E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \varepsilon > \varepsilon_y$

$$\sigma_y = \sigma_1 + E_1\varepsilon_y$$

Power law equations $\quad \sigma = A\varepsilon^n \quad E_{sec} = \frac{\sigma_y + A\varepsilon^n}{\varepsilon_T} \qquad \sigma \geq \sigma_y$

$$\varepsilon_T = \sqrt[n]{\left(\frac{\sigma_{eff} - \sigma_y}{A}\right) + \varepsilon_y}$$

Ramberg-Osgood equation$\varepsilon = \frac{\sigma}{E}\left(1 + 0.002\left(\frac{\sigma}{\sigma_y}\right)^{n-1}\right) \qquad \frac{1}{E_{sec}} = \frac{1}{E}\left(1 + \frac{0.3}{0.7}\left(\frac{\sigma}{\sigma_{0.7}}\right)^{n-1}\right)$ for aluminum

Most deformation processes involving thin plates of material are approximated to the plane stress conditions

Plane stress is a state of stress in which the normal stress σ_z, and the shear stresses σ_{xz}, σ_{yz} directed perpendicular to the x-y plane are assumed to be zero

The geometry of the body is that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown.

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

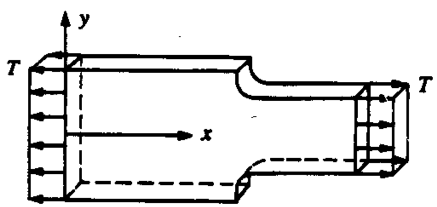


plate with fillet

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

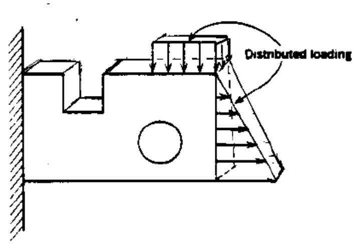


plate with hole

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

The plane stress condition is the simplest form of behavior for continuum structures and represents situations frequently encountered in practice

The effective stress resulting from loading of the material is calculated as

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

For plane stress condition

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y\right)^2 + (-\sigma_x)^2 + 6\left(\tau_{xy}{}^2\right)}$$

The secant modulus will be obtained from the material model equations and the corresponding poisson's ratio and shear modulus will be calculated using

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

$$G^p = \frac{E_{sec}}{2(1+\nu)}$$

Loading state on the aluminum sheet is $F_x = 70000\ kN,\ F_y = 0\ kN,\ F_{xy} = 40000\ kN$

The effective stress

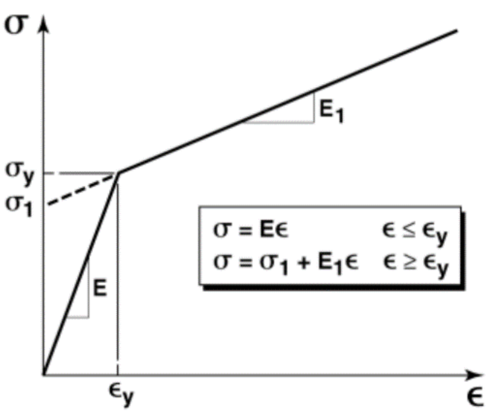
$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(70-0)^2 + (0)^2 + (70)^2 + 6(40^2)} = 98.5\ MPa$$

Substitute the effective stress into the material model equation

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma_y = \sigma_1 + E_1\varepsilon_y \qquad \varepsilon > \varepsilon_y$$

$\sigma_1 = 30 - 5000 * 0.00043 = 27.85$ MPa

$$\varepsilon = \frac{98.5 - 27.85}{5000} = 0.0141$$



The secant modulus, plastic shear modulus and the poisons ratio become

$E_{sec} = \frac{27.85}{0.0141} + 5000 = 6971$ MPa

$$\nu = \frac{1}{2} - \frac{6971}{70000}\left(\frac{1}{2} - 0.35\right) = 0.485$$

$G^p = \frac{6971}{2(1+0.485)} = 2347$ MPa

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{6971}\big(70 - 0.485(0+0)\big) = 0.01$$

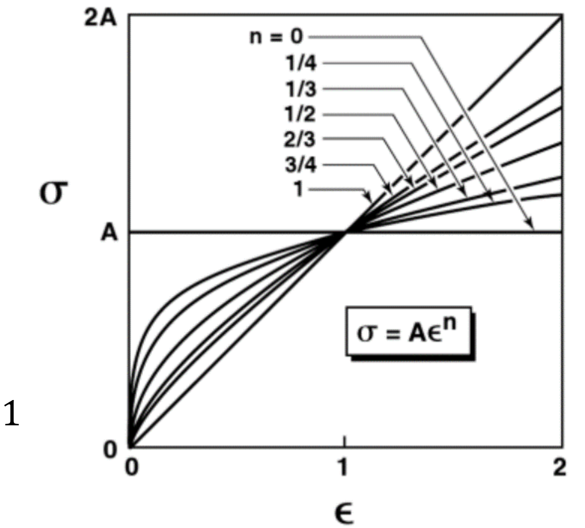
$$\varepsilon_y = \frac{1}{6971}\big(0 - 0.485(70+0)\big) = -0.00487$$

$$\varepsilon_z = \frac{1}{6971}\big(0 - 0.485(70+0)\big) = -0.00487$$

$$\gamma_{xy} = \frac{40}{2347} = 0.017$$

For the power law material model the secant modulus is obtained as

$$E_{sec} = \frac{30+500\varepsilon^{0.5}}{0.01919} = 5173 \text{ MPa} \qquad \sigma \geq 30 \text{ Mpa}$$



$$\varepsilon = \sqrt[0.5]{\left(\frac{98.5-30}{500}\right)} + 0.00043 = 0.0191$$

The shear modulus and the poisson's ratio become

$$\nu = \frac{1}{2} - \frac{5173}{70000}\left(\frac{1}{2} - 0.35\right) = 0.474$$
$$G^p = \frac{5173}{2(1+0.474)} = 1754.75$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{5173}\big(70 - 0.474(0+0)\big) = 0.0135$$
$$\varepsilon_y = \frac{1}{5173}\big(0 - 0.474(70+0)\big) = -0.0066$$
$$\varepsilon_z = \frac{1}{5173}\big(0 - 0.474(70+0)\big) = -0.0066$$
$$\gamma_{xy} = \frac{40}{1754.75} = 0.023$$

For the Ramberg-Osgood model the secant modulus becomes

$$\frac{1}{E_{sec}} = \frac{1}{70000}\left(1 + \frac{0.3}{0.7}\left(\frac{98.5}{40}\right)^{5-1}\right),\ E_{sec} = 4178.7$$

$$\nu = \frac{1}{2} - \frac{4179}{E}\left(\frac{1}{2} - 0.35\right) = 0.491$$

The shear modulus and the poisons ratio become

$$G^p = \frac{4179}{2(1+0.491)} = 1401$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{4179}\big(70 - 0.491(20+0)\big) = 0.01675$$
$$\varepsilon_y = \frac{1}{4179}\big(20 - 0.491(70+0)\big) = -0.00823$$
$$\varepsilon_z = \frac{1}{4179}\big(0 - 0.491(70+20)\big) = -0.00823$$
$$\gamma_{xy} = \frac{40}{1401} = 0.0286$$

Comparing the results for the three models:

	ε_x	ε_y	ε_z	γ_{xy}
Linear hardening strains	0.01	-0.00487	-0.00487	0.017
Power law	0.0135	-0.0066	-0.0066	0.023
Ramberg-Osgood	0.01675	-0.00823	-0.00823	0.0286

When the loading state on the aluminum sheet is $F_x = 70000\ kN,\ F_y = 20000\ kN,\ F_{xy} = 40000$ kN
The effective stress

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(70-\sigma_y\right)^2 + (20)^2 + (70)^2 + 6(40^2)} = 93.27\ MPa$$

Substitute the effective stress into the material model equation

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma_y = \sigma_1 + E_1\varepsilon_y \qquad \varepsilon > \varepsilon_y$$

$\sigma_1 = 30 - 5000 * 0.00043 = 27.85$ MPa

$$\varepsilon = \frac{93.27 - 27.85}{5000} = 0.0131$$

The secant modulus, plastic shear modulus and the poisons ratio become

$E_{sec} = \frac{27.85}{0.0131} + 5000 = 7128$ MPa

$$\nu = \frac{1}{2} - \frac{7128}{70000}\left(\frac{1}{2} - 0.35\right) = 0.4847$$

$G^p = \frac{7128}{2(1+0.4847)} = 2400$ MPa

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{7128}\left(70 - 0.4847(20+0)\right) = 0.00846$$

$$\varepsilon_y = \frac{1}{7128}\left(20 - 0.4847(70+0)\right) = -0.00195$$

$$\varepsilon_z = \frac{1}{7128}\left(0 - 0.4847(70+20)\right) = -0.00612$$

$$\gamma_{xy} = \frac{40}{2400} = 0.0167$$

For the power law material model the secant modulus is obtained as

$$E_{sec} = \frac{30+500\varepsilon^{0.5}}{0.01644} = 4904 \text{ MPa} \qquad \sigma \geq 30 \text{ MPa}$$

$$\varepsilon = \sqrt[0.5]{\left(\frac{93.274-30}{500}\right)} + 0.00043 = 0.01644$$

The shear modulus and the poisons ratio become

$$\nu = \frac{1}{2} - \frac{4904}{70000}\left(\frac{1}{2} - 0.35\right) = 0.4895$$

$$G^p = \frac{5173}{2(1+0.474)} = 1662 \text{ MPa}$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{4904}\big(70 - 0.4755(20+0)\big) = 0.0123$$

$$\varepsilon_y = \frac{1}{4904}\big(20 - 0.4755(70+0)\big) = -0.0027$$

$$\varepsilon_z = \frac{1}{4904}\big(0 - 0.4755(70+20)\big) = -0.0087$$

$$\gamma_{xy} = \frac{40}{1662} = 0.02407$$

For the Ramberg-Osgood model the secant modulus becomes

$$\frac{1}{E_{sec}} = \frac{1}{70000}\left(1+\frac{0.3}{0.7}\left(\frac{93.274}{40}\right)^{5-1}\right),\ E_{sec} = 5120$$

$$\nu = \frac{1}{2} - \frac{5120}{E}\left(\frac{1}{2} - 0.35\right) = 0.489$$

The shear modulus and the poisons ratio become

$$G^p = \frac{5120}{2(1+0.489)} = 1719$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{5120}\big(70 - 0.489(20+0)\big) = 0.01176$$

$$\varepsilon_y = \frac{1}{5120}\big(20 - 0.489(70+0)\big) = -0.00278$$

$$\varepsilon_z = \frac{1}{5120}\big(0 - 0.489(70+20)\big) = -0.0086$$

$$\gamma_{xy} = \frac{40}{1719} = 0.0233$$

Comparing the results for the three models:

	ε_x	ε_y	ε_z	γ_{xy}
Linear hardening strains	0.00846	-0.00195	-0.00612	0.0167
Power law	0.0123	-0.0027	-0.0087	0.023
Ramberg-Osgood	0.01176	-0.00278	0.0086	0.0233

Consider a multiaxial stress state on the same material with all 6 normal and shear stresses

$\sigma_x = 70\ MPa, \quad \sigma_y = 30\ MPa, \quad \sigma_z = 50\ MPa, \quad \tau_{xy} = 40\ MPa, \quad \tau_{yz} = 30\ MPa \quad \tau_{zx} = 10\ MPa$

All normal forces are tensile

The effective stress

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(70-30)^2 + (30-50)^2 + (50-70)^2 + 6(40^2 + 30^2 + 10^2)} = 94.87\ MPa$$

Substitute the effective stress into the material model equation

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma_y = \sigma_1 + E_1\varepsilon_y \qquad \varepsilon > \varepsilon_y$$

$$\sigma_1 = 30 - 5000 * 0.00043 = 27.85 \text{ MPa}$$

$$\varepsilon = \frac{94.87 - 27.85}{5000} = 0.0134$$

The secant modulus, plastic shear modulus and the poisons ratio become

$$E_{sec} = \frac{27.85}{0.0134} + 5000 = 7078 \text{ MPa}$$

$$\nu = \frac{1}{2} - \frac{7078}{70000}\left(\frac{1}{2} - 0.35\right) = 0.4848$$

$$G^p = \frac{7078}{2(1+0.4848)} = 2383 \text{ MPa}$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{7078}\left(70 - 0.4848(30+50)\right) = 0.0044$$

$$\varepsilon_y = \frac{1}{7078}\left(30 - 0.4848(70+50)\right) = -0.004$$

$$\varepsilon_z = \frac{1}{7078}\left(50 - 0.4848(70+30)\right) = 0.000214$$

$$\gamma_{xy} = \frac{40}{2383} = 0.0168 \quad \gamma_{yz} = \frac{30}{2383} = 0.0126 \quad \gamma_{zx} = \frac{10}{2383} = 0.0042$$

For the power law material model the secant modulus is obtained as

$$E_{sec} = \frac{30+500\varepsilon^{0.5}}{0.0177} = 4986.4 \text{ MPa} \qquad \sigma \geq 30 \text{ MPa}$$

$$\varepsilon = \sqrt[0.5]{\left(\frac{94.87-30}{500}\right)} + 0.00043 = 0.01726$$

The shear modulus and the poisons ratio become

$$\nu = \frac{1}{2} - \frac{4986.4}{70000}\left(\frac{1}{2} - 0.35\right) = 0.4893$$

$$G^{p} = \frac{4986.4}{2(1+0.489)} = 1674 \text{ Mpa}$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{4986}\big(70 - 0.489(30+50)\big) = 0.00619$$

$$\varepsilon_y = \frac{1}{4986}\big(30 - 0.489(70+50)\big) = -0.00576$$

$$\varepsilon_z = \frac{1}{4986}\big(50 - 0.489(70+30)\big) = -0.00021$$

$$\gamma_{xy} = \frac{40}{1674} = 0.0239$$

$$\gamma_{yz} = \frac{30}{1674} = 0.0179$$

$$\gamma_{zx} = \frac{10}{1674} = 0.006$$

For the Ramberg-Osgood model the secant modulus becomes

$$\frac{1}{E_{sec}} = \frac{1}{70000}\left(1 + \frac{0.3}{0.7}\left(\frac{94.7}{40}\right)^{5-1}\right), \; E_{sec} = 4807.6$$

The shear modulus and the poisons ratio become

$$G^p = \frac{4808}{2(1+0.49)} = 1619$$

$$\nu = \frac{1}{2} - \frac{4808}{70000}\left(\frac{1}{2} - 0.35\right) = 0.49$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{4808}\big(70 - 0.49(30 - 50)\big) = 0.0064$$

$$\varepsilon_y = \frac{1}{4808}\big(30 - 0.49(70 - 50)\big) = -0.006$$

$$\varepsilon_z = \frac{1}{4808}\big(50 - 0.49(70 - 50)\big) = 0.00021$$

$$\gamma_{xy} = \frac{40}{1619} = 0.0247$$

$$\gamma_{yz} = \frac{30}{1619} = 0.0185$$

$$\gamma_{zx} = \frac{10}{1619} = 0.0062$$

Comparing the results for the three models:

	ε_x	γ_{xy}	ε_y	γ_{yz}	ε_z	γ_{zx}
Linear hardening strains	0.0044	0.0168	-0.004	0.0126	0.0002	0.0042
Power law	0.0062	0.024	-0.0058	0.018	-0.0002	0.006
Ramberg-Osgood	0.0064	0.0247	-0.006	0.0185	0.00021	0.0062

For the case of tensile and compressive normal stresses the deformations change

$$\sigma_x = 20\ MPa, \qquad \sigma_y = -30\ MPa, \qquad \sigma_z = -10\ MPa, \qquad \tau_{xy} = 40\ MPa \quad \tau_{yz} = 30\ MPa \quad \tau_{zx} = 10\ MPa$$

The effective stress

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(20 - -30)^2 + (-30 - -10)^2 + (-10 - 20)^2 + 6(40^2 + 30^2 + 10^2)} = 98.5\ MPa$$

Substitute the effective stress into the material model equation

$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma_y = \sigma_1 + E_1\varepsilon_y \qquad \varepsilon > \varepsilon_y$$

$\sigma_1 = 30 - 5000 * 0.00043 = 27.85$ MPa

$$\varepsilon = \frac{98.5 - 27.85}{5000} = 0.01413$$

The secant modulus, plastic shear modulus and the poisons ratio become

$E_{sec} = \frac{27.85}{0.01413} + 5000 = 6971$ MPa

$$\nu = \frac{1}{2} - \frac{6971}{70000}\left(\frac{1}{2} - 0.35\right) = 0.485$$

$G^p = \frac{6971}{2(1+0.485)} = 2347$ Mpa

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{6971}\big(20 - 0.485(-30 - 10)\big) = 0.0057$$

$$\varepsilon_y = \frac{1}{6971}\big(-30 - 0.485(20 - 10)\big) = -0.005$$

$$\varepsilon_z = \frac{1}{6971}\big(-10 - 0.485(20 - 30)\big) = -0.00074$$

$$\gamma_{xy} = \frac{40}{2347} = 0.017 \qquad \gamma_{yz} = \frac{30}{2347} = 0.0128 \qquad \gamma_{zx} = \frac{10}{2347} = 0.00426$$

For the power law material model the secant modulus is obtained as

$$E_{sec} = \frac{30+500\varepsilon^{0.5}}{0.0196} = 5172.8 \text{ MPa} \qquad \sigma \geq 30 \text{ MPa}$$

$$\varepsilon = \sqrt[0.5]{\left(\frac{98.5-30}{500}\right)} + 0.00043 = 0.0192$$

The shear modulus and the poisons ratio become

$$\nu = \frac{1}{2} - \frac{5173}{70000}\left(\frac{1}{2} - 0.35\right) = 0.489$$

$$G^p = \frac{4986.4}{2(1+0.489)} = 1737 \text{ MPa}$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{5173}\big(20 - 0.489(-30-10)\big) = 0.007647$$

$$\varepsilon_y = \frac{1}{5173}\big(-30 - 0.489(20-10)\big) = -0.00674$$

$$\varepsilon_z = \frac{1}{5173}\big(-10 - 0.489(20-30)\big) = -0.001$$

$$\gamma_{xy} = \frac{40}{1737} = 0.023$$

$$\gamma_{yz} = \frac{30}{1737} = 0.0173$$

$$\gamma_{zx} = \frac{10}{1737} = 0.0058$$

For the Ramberg-Osgood model the secant modulus becomes

$$\frac{1}{E_{sec}} = \frac{1}{70000}\left(1 + \frac{0.3}{0.7}\left(\frac{98.5}{40}\right)^{5-1}\right),\ E_{sec} = 4178.7$$

$$\nu = \frac{1}{2} - \frac{4179}{70000}\left(\frac{1}{2} - 0.35\right) = 0.491$$

The shear modulus and the poisons ratio become

$$G^p = \frac{4179}{2(1+0.491)} = 1407$$

The strains are accordingly calculated as

$$\varepsilon_x = \frac{1}{4179}\big(20 - 0.491(-30 - 10)\big) = 0.0095$$

$$\varepsilon_y = \frac{1}{4179}\big(-30 - 0.491(20 - 10)\big) = -0.0084$$

$$\varepsilon_z = \frac{1}{4179}\big(-10 - 0.491(20 - 30)\big) = -0.00122$$

$$\gamma_{xy} = \frac{40}{1407} = 0.0284 \quad \gamma_{yz} = \frac{30}{1407} = 0.0213 \quad \gamma_{zx} = \frac{10}{1407} = 0.0071$$

	ε_x	γ_{xy}	ε_y	γ_{yz}	ε_z	γ_{zx}
Linear hardening strains	0.0057		-0.005		-0.00074	
		0.017		0.0128		0.00426
Power law	0.00765		-0.0067		-0.001	
		0.023		0.0173		0.0058
Ramberg-Osgood	0.0095		-0.0084		-0.0012	
		0.0284		0.0213		0.0071

Overall comparison

Stress state $F_x = 70000\ kN,\ \sigma_y = 0\ kN,\ \tau_{xy} = 40000\ kN$

	ε_x	ε_y	ε_z	γ_{xy}
Linear hardening strains	0.01	-0.00487	-0.00487	0.017
Power law	0.0135	-0.0066	-0.0066	0.023
Ramberg-Osgood	0.01675	-0.00823	-0.00823	0.0286

$F_x = 70000\ kN,\ F_y = 20000\ kN,\ F_{xy} = 40000$ kN

	ε_x	ε_y	ε_z	γ_{xy}
Linear hardening strains	0.00846	-0.00195	-0.00612	0.0167
Power law	0.0123	-0.0027	-0.0087	0.023
Ramberg-Osgood	0.01176	-0.00278	0.0086	0.0233

$\sigma_x = 70\ MPa,\quad \sigma_y = 30\ MPa,\quad \sigma_z = 50\ MPa,\quad \tau_{xy} = 40\ MPa,\quad \tau_{yz} = 30\ MPa\quad \tau_{zx} = 10\ MPa$

	ε_x	γ_{xy}	ε_y	γ_{yz}	ε_z	γ_{zx}
Linear hardening strains	0.0044		-0.004		0.0002	
		0.0168		0.0126		0.0042
Power law	0.0062		-0.0058		-0.0002	
		0.024		0.018		0.006
Ramberg-Osgood	0.0064		-0.006		0.00021	
		0.0247		0.0185		0.0062

$\sigma_x = 20\ MPa,\ \sigma_y = -30\ MPa,\ \sigma_z = -10\ MPa,\ \tau_{xy} = 40\ MPa\ \ \tau_{yz} = 30\ MPa\ \ \tau_{zx} = 10\ MPa$

	ε_x	γ_{xy}	ε_y	γ_{yz}	ε_z	γ_{zx}
Linear hardening strains	0.0057		-0.005		-0.00074	
		0.017		0.0128		0.00426
Power law	0.00765		-0.0067		-0.001	
		0.023		0.0173		0.0058
Ramberg-Osgood	0.0095		-0.0084		-0.0012	
		0.0284		0.0213		0.0071

Exercise Question

An aluminum rectangular prism, with sides 60x80x100 cm long and that are oriented parallel to the the x, y, z axes, is subjected to normal forces in three dimensions with F_x= 80000 kN, F_y= -12000 kN, F_z= -24000 kN. The material undergoes plastic deformation as its yield strength is 30 MPa and E= 70GPa, ν= 0.35. The stress-strain curve of aluminum is approximated by the power law model with the equation $\sigma = 500 * \varepsilon^{0.5}$